About two properties of Fibonacci Numbers
(Or, Common solution of two problems with Fibonacci Numbers)

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Here we consider following two problems **.**

Problem1.

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Show that the difference of squares of Fibonacci numbers whose positions in the sequence differ by two, is again a Fibonacci number.

and

Problem 2.

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Show that the sum of the squares of two consecutive Fibonacci numbers is again a Fibonacci number.

Both these problems related to Fibonacci Numbers f_n, defined recursively by

$$f_{n+1} = f_n + f_{n-1}, n \in \mathbb{N} \text{ and } f_0 = 0, f_1 = 1,$$

represents two identities

$$f_n^2 + f_{n+1}^2 = f_{2n+1} \text{ and } f_{n+1}^2 - f_{n-1}^2 = f_{2n} \text{ which holds for any } n \in \mathbb{N}.$$

The following four proofs of these identities represents different approaches to solve both problems.

Proof 1.

Applying Luca's Formula* $f_{n+m} = f_{m-1}f_n + f_m f_{n+1}$ for $m = n$ and $m = n+1$ we obtain, respectively,

$$f_{2n} = f_{n-1}f_n + f_n f_{n+1} = (f_{n+1} + f_{n-1})(f_{n+1} - f_{n-1}) = f_{n+1}^2 - f_{n-1}^2 \; and \; f_{2n+1} = f_n^2 + f_{n+1}^2$$

.

* For any fixed $m \in \mathbb{N} \cup \{0\}$we will find representation of f_{n+m} as linear combination of f_n and f_{n+1}that is in the form $f_{n+m} = \alpha_m f_n + \beta_m f_{n+1}$.

Then we have $\alpha_0 = 1, \beta_0 = 0$ (since $f_{n+0} = 1 \cdot f_n + 0 \cdot f_{n+1}$), $\alpha_1 = 0, \beta_1 = 1$ (since $f_{n+1} = 0 \cdot f_n + 1 \cdot f_{n+1}$) and $\alpha_{n+1} = \alpha_n + \alpha_{n-1}, \beta_{n+1} = \beta_n + \beta_{n-1}, n \in \mathbb{N}$ (since $f_{n+m+1} = f_{n+m} + f_{n+m-1} \iff \alpha_{m+1}f_n + \beta_{m+1}f_{n+1} = \alpha_m f_n + \beta_m f_{n+1} + \alpha_{m-1}f_n + \beta_{m-1}f_{n+1} \iff$

$$\alpha_{m+1}f_n + \beta_{m+1}f_{n+1} = (\alpha_m + \alpha_{m-1}) f_n + (\beta_m + \beta_{m-1}) f_{n+1}).$$

Taking in account that $f_{-1} = f_1 - f_0 = 1 - 0 = 1$ by Math Induction we obtain that $\alpha_m = f_{m-1}$ and $\beta_m = f_m$ for any $m \in \mathbb{N} \cup \{0\}$.

Proof 2.

Since $f_{n+2} = f_{n+1} + f_n$, $f_{n+1} = f_n + f_{n-1}$ and $f_n - f_{n-1} = f_{n-2}$ then

$$f_{n+2} = 2f_n + f_{n-1} = 3f_n + f_{n-1} - f_n = 3f_n - f_{n-2}.$$

Hence, $f_{n+2} = 3f_n - f_{n-2}$ and, therefore,

$$f_{2n+2} = 3f_{2n} - f_{2n-2}, \quad f_{2n+3} = 3f_{2n+1} - f_{2n-1}, n \in \mathbb{N}.$$

Note that

$$\left(f_{n+2}^2 - f_n^2\right) + \left(f_n^2 - f_{n-2}^2\right) = f_{n+2}^2 - f_{n-2}^2 = \left(f_{n+1} + f_n\right)^2 - \left(f_n - f_{n-1}\right)^2 =$$

$$\left(f_{n+1} + f_{n-1}\right)\left(2f_n + f_{n+1} - f_{n-1}\right) = 3\left(f_{n+1} - f_{n-1}\right)\left(f_{n+1} + f_{n-1}\right) = 3\left(f_{n+1}^2 - f_{n-1}^2\right)$$

and

$$f_2^2 - f_0^2 = 1 - 0 = 1 = f_2,\ f_1^2 - f_{-1}^2 = 1 - 1 = 0 = f_0.$$

Then, since both sequences $f_{n+1}^2 - f_{n-1}^2$ and f_{2n} satisfies to the same recurrence and the same initial values for $n = 0, 1$ we can conclude, using Math Induction, that $f_{n+1}^2 - f_{n-1}^2 = f_{2n}$.

Similarly, since

$$f_0^2 + f_{0+1}^2 = 1 = f_1,\ f_1^2 + f_{1+1}^2 = 2 = f_3$$

and

$$\left(f_{n+1}^2 + f_{n+2}^2\right) + \left(f_n^2 + f_{n-1}^2\right) = f_{n+1}^2 + \left(f_{n+1} + f_n\right)^2 + f_n^2 + \left(f_{n+1} - f_n\right)^2 = 3\left(f_n^2 + f_{n+1}^2\right)$$

then then by Math Induction $f_n^2 + f_{n+1}^2 = f_{2n+1}$ for any $n \in \mathbb{N}$

Proof 3.Math Induction.

We will prove,using Math Induction, that

$$f_n^2 + f_{n+1}^2 = f_{2n+1} \text{ and } f_{n+1}^2 - f_{n-1}^2 = f_{2n}$$

holds for any $n \in \mathbb{N}$.

Note that $f_{n+1}^2 - f_{n-1}^2 = \left(f_{n+1} - f_{n-1}\right)\left(f_{n+1} + f_{n-1}\right) = f_n\left(f_{n+1} + f_{n-1}\right)$.

Step of Math Induction:

For any $n \in \mathbb{N}$ assuming $f_{2n-1} = f_{n-1}^2 + f_n^2$ and $f_{n+1}^2 - f_{n-1}^2 = f_{2n}$ we obtain

$$f_{2n+1} = f_{2n} + f_{2n-1} = f_{n+1}^2 - f_{n-1}^2 + f_{n-1}^2 + f_n^2 = f_n^2 + f_{n+1}^2.$$

$$f_{2n+2} = f_{2n+1} + f_{2n} = f_n^2 + f_{n+1}^2 + f_n\left(f_{n+1} + f_{n-1}\right) = f_n^2 + f_{n+1}^2 + f_n f_{n+1} + f_n f_{n-1} =$$

$$f_n^2 + f_n f_{n-1} + f_n f_{n+1} + f_{n+1}^2 = f_n\left(f_n + f_{n-1}\right) + f_{n+1}\left(f_n + f_{n+1}\right) =$$
$$f_n f_{n+1} + f_{n+1} f_{n+2} = f_{n+1}\left(f_n + f_{n+2}\right) = f_{n+2}^2 - f_n^2.$$

Proof 4.. (With Bine't formula).

Since $f_n = \dfrac{\phi^n - \overline{\phi}^n}{\phi - \overline{\phi}}$ then $f_n^2 + f_{n+1}^2 = \dfrac{\left(\phi^n - \overline{\phi}^n\right)^2}{\left(\phi - \overline{\phi}\right)^2} + \dfrac{\left(\phi^{n+1} - \overline{\phi}^{n+1}\right)^2}{\left(\phi - \overline{\phi}\right)^2} =$

$$\frac{\phi^{2n} + \overline{\phi}^{2n} - 2\left(\phi\overline{\phi}\right)^n + \phi^{2(n+1)} + \overline{\phi}^{2(n+1)} - 2\left(\phi\overline{\phi}\right)^{n+1}}{\left(\phi - \overline{\phi}\right)^2} = \frac{\phi^{2n} + \overline{\phi}^{2n} + \phi^{2(n+1)} + \overline{\phi}^{2(n+1)}}{\left(\phi - \overline{\phi}\right)^2}$$

Since $\phi\overline{\phi} = -1$ we have

$$\phi^{2n} + \phi^{2(n+1)} + \overline{\phi}^{2n} + \overline{\phi}^{2(n+1)} = \phi^{2n+1}\left(\phi^{-1} + \phi\right) + \overline{\phi}^{2n+1}\left(\overline{\phi}^{-1} + \overline{\phi}\right) =$$

$$\phi^{2n+1}\left(\phi - \overline{\phi}\right) + \overline{\phi}^{2n+1}\left(\overline{\phi} - \phi\right) = \left(\phi^{2n+1} - \overline{\phi}^{2n+1}\right)\left(\phi - \overline{\phi}\right).$$

Hence, $f_n^2 + f_{n+1}^2 = \dfrac{\phi^{2n+1} - \overline{\phi}^{2n+1}}{\phi - \overline{\phi}} = f_{2n+1}.$

Also,

$$f_{n+1}^2 - f_{n-1}^2 = \frac{\left(\phi^{n+1} - \overline{\phi}^{n+1}\right)^2}{\left(\phi - \overline{\phi}\right)^2} - \frac{\left(\phi^{n-1} - \overline{\phi}^{n-1}\right)^2}{\left(\phi - \overline{\phi}\right)^2} =$$

$$\frac{\phi^{2n+2} + \overline{\phi}^{2n+2} - 2\left(\phi\overline{\phi}\right)^{n+1} - \phi^{2n-2} - \overline{\phi}^{2n-2} + 2\left(\phi\overline{\phi}\right)^{n-1}}{\left(\phi - \overline{\phi}\right)^2} = \frac{\phi^{2n+2} + \overline{\phi}^{2n+2} - \phi^{2n-2} - \overline{\phi}^{2n-2}}{\left(\phi - \overline{\phi}\right)^2}$$

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Since $\phi^2\overline{\phi}^2 = 1$ and $\phi + \overline{\phi} = 1$ then

$$f_{n+1}^2 - f_{n-1}^2 = \frac{\phi^{2n+2} + \overline{\phi}^{2n+2} - \phi^{2n-2} - \overline{\phi}^{2n-2}}{\left(\phi - \overline{\phi}\right)^2} = \frac{\phi^{2n}\left(\phi^2 - \overline{\phi}^2\right) + \overline{\phi}^{2n}\left(\overline{\phi}^2 - \phi^2\right)}{\left(\phi - \overline{\phi}\right)^2} =$$

$$\frac{\left(\phi^2 - \overline{\phi}^2\right)\left(\phi^{2n} - \overline{\phi}^{2n}\right)}{\left(\phi - \overline{\phi}\right)^2} = \frac{\phi^{2n} - \overline{\phi}^{2n}}{\phi - \overline{\phi}} = f_{2n}.$$

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